HILBERT MODULAR SURFACES, REVISITED

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1. Motivation - modular curves

Let $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{im} z > 0\}$ be the complex upper half plane. It admits a natural action of $\mathrm{GL}_2^+(\mathbb{R})$. For a discrete subgroup $\Gamma \le \mathrm{GL}_2^+(\mathbb{R})$ we can consider the quotient $Y_\Gamma = \Gamma\backslash\mathcal{H}$, which admits a natural compactification $X_\Gamma = Y_\Gamma \cup \{\Gamma\backslash\mathbb{P}^1(\mathbb{Q})\}$. As an example, if $\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \mathrm{GL}_2^+(\mathbb{Z})$, then $\Gamma\backslash\mathcal{H}$ is the well-known fundamental domain D (draw), and X_Γ is isomorphic to the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. In this case, X_Γ parametrizes isomorphism classes of elliptic curves (or of homothety classes of lattices), through the map

$$\tau \mapsto \Lambda_\tau = \mathbb{Z}\cdot\tau + \mathbb{Z}\cdot 1 \mapsto E_\tau = \mathbb{C}/\Lambda_\tau.$$

More generally, if Γ is a congruence subgroup, X_Γ is a Riemann surface, parametrizing isomorphism classes of elliptic curves with additional data. In fact, X_Γ admits the structure of an algebraic curve over a number field, and questions about elliptic curves translate to studying the properties of these modular curves.

Important examples include $X_0(N) = X_{\Gamma_0(N)}$ parametrizing elliptic curves with a cyclic N-isogeny, and $X_1(N) = X_{\Gamma_1(N)}$ parametrizing elliptic curves with an N-torsion point.

From the genus of a modular curve, using Faltings's theorem, we can deduce that if $g(X_\Gamma) > 1$, there are finitely many rational points on X_Γ, corresponding to finitely many elliptic curves with this level structure (e.g. elliptic curves with rational N-torsion), and if $g(X_\Gamma) = 0$ and X_Γ has a point, then there are infinitely many such. An example application is Mazur's celebrated theorem on the classification of rational torsion on elliptic curves.

Recently, due to advances in algorithms for computing equations for these curves X_Γ, there has been a systematic effort to create a database of these modular curves. (reference beta version of the LMFDB).

In a similar vein, one can consider moduli spaces of abelian surfaces with (potential) QM, which also form curves, and study their geometry. This leads to corresponding results, as in recent work of Laga-Schembri-Shnidman-Voight (2023) classifying possible torsion on such surfaces.

2. Hilbert modular varieties

This generalizes (in more than one way) to higher dimensions. One way is to consider a totally real number field F of degree n, with ring of integers $\mathbb{Z}_F$, and the group $\mathrm{GL}_2^+(F)$ of matrices with totally positive determinant. Then through its n real embeddings, this group acts on $\mathcal{H}^n$. Again, by considering a discrete subgroup Γ, we may form the quotient $\Gamma\backslash\mathcal{H}^n$, and compactify it by adjoining the cusps $\Gamma\backslash\mathbb{P}^1(F) \hookrightarrow \mathbb{P}^1(\mathbb{R})^n$, to obtain a compact Hilbert modular variety $Y_\Gamma = (\Gamma\backslash\mathcal{H}^n)^*$.

Trying to generalize the standard example from before, we already run into several differences from the modular curve case.

2.1. **Choice of arithmetic subgroup.** The first is that $\mathrm{SL}_2(\mathbb{Z}_F) \neq \mathrm{GL}_2^+(\mathbb{Z}_F)$, as there are non-trivial totally positive units in our field. In fact, if $\mathbb{Z}_{F,>0}^\times \neq \mathbb{Z}_F^{\times 2}$, then even $\mathrm{PSL}_2(\mathbb{Z}_F) \neq \mathrm{PGL}_2^+(\mathbb{Z}_F)$, yielding different quotients of $\mathcal{H}^n$.

Example 2.1. Let $F = \mathbb{Q}(\sqrt{3})$. Then $\mathbb{Z}_F^\times = \{\pm 1\} \times \epsilon^{\mathbb{Z}}$, where $\epsilon = 2 - \sqrt{3} \in \mathbb{Z}_{F,>0}^\times$ is totally positive. In particular, the matrix

$$\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Z}_F)$$

maps to an element in $\mathrm{PGL}_2^+(\mathbb{Z}_F)$, which is not in the image of $\mathrm{SL}_2(\mathbb{Z}_F)$.

2.2. **Moduli space.** The second concerns the moduli interpretation. Given a point $\tau = (\tau_1, \ldots, \tau_n) \in \mathcal{H}^n$, we may consider the lattice

$$\Lambda_\tau = \mathbb{Z}_F \cdot \tau + \mathbb{Z}_F \cdot 1 = \{(a_1\tau_1 + b_1, \ldots, a_n\tau_n + b_n) : a, b \in \mathbb{Z}_F\} \subseteq \mathbb{C}^n,$$

where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are images of a, b, respectively, under the the different real embeddings. Then $A_\tau = \mathbb{C}^n/\Lambda_\tau$ is a complex torus, with real multiplication by $\mathbb{Z}_F$ through the diagonal map. Recall that a complex torus is an abelian variety if and only if it admits an ample line bundle (equivalently, a polarization), and line bundles on A_τ (through their Chern classes) are in bijection with Riemann forms on $\mathbb{C}^n$ with respect to Λ_τ. Any such form which is F-linear must be of the form

$$H_{r,\tau}(x, y) = \sum \frac{x_i \overline{y_i} r_i}{\operatorname{im} \tau_i}, \quad r \in \mathfrak{d}_F^{-1},$$

and it is positive definite (corresponds to an ample line bundle) if and only if r is totally positive. Therefore $\mathrm{GL}_2^+(\mathbb{Z}_F)\backslash\mathcal{H}^n$ parametrizes isomorphism classes of abelian surfaces with real multiplication by $\mathbb{Z}_F$ and polarization cone $\mathfrak{d}_{F,>0}^{-1}$. If we wish to parametrize all abelian surfaces with real multiplication by $\mathbb{Z}_F$, (and allow for other polarizations), we really should be asking more generally about groups of the form $\mathrm{GL}^+(\mathbb{Z}_F \oplus \mathfrak{b})$, where $\mathfrak{b}$ is a fractional ideal of $\mathbb{Z}_F$. Since multiplying by a totally positive element does not change the isomorphism class, $\mathfrak{b}$ could be taken from a set of representatives of the narrow class group. We could also specify the polarization, which amounts to considering SL_2 instead of GL_2^+.

Example 2.2. Continue with $F = \mathbb{Q}(\sqrt{3})$. As $\mathrm{Cl}^+(F) \simeq \mathbb{Z}/2\mathbb{Z}$, with representatives being $\mathbb{Z}_F, \mathfrak{p}_3 = (\sqrt{3})$, the moduli space of (isomorphism classes of) abelian varieties with real multiplication by $\sqrt{3}$ has two components, $X(\mathbb{Z}_F)_{\mathbb{Z}_F} = \mathrm{GL}_2^+(\mathbb{Z}_F)\backslash\mathcal{H}^2$ and $X(\mathbb{Z}_F)_{\mathfrak{p}_3} = \mathrm{GL}^+(\mathbb{Z}_F \oplus \mathfrak{p}_3)\backslash\mathcal{H}^2$ corresponding to the two possible polarization modules. Note that the latter corresponds to the polarization module being $\mathbb{Z}_F$, hence the component of principally polarizable abelian varieties with real multiplication by $\sqrt{3}$. (This one could also be identified with $\mathrm{GL}_2^+(\mathbb{Z}_F)\backslash(\mathcal{H}^+ \times \mathcal{H}_-)$). Looking at the SL_2 variants we obtain the moduli space of principally polarized abelian varieties with real multiplication by $\sqrt{3}$ (equipped with the polarization).

2.3. **Singularities.** The third is that these surfaces are singular at cusps and elliptic points (as these are "codimension n corners"). Looking at the smallest example, we focus on the case $n = 2$. Then there exists a minimal desingularization of the complex surface Y_Γ, which we denote by X_Γ. Hirzebruch (1970's), with collaborators (Van der Geer, Van de Ven, Zagier among others) investigated the Hilbert modular surfaces $\mathrm{SL}(\mathbb{Z}_F \oplus \mathfrak{b})\backslash\mathcal{H}^2$ and constructed a resolution of singularities at the cusps and at the elliptic points of these surfaces $X^1(\mathbb{Z}_F)_{\mathfrak{b}}$.

But what replaces the genus now that we have a surface?

3. Geometric invariants of surfaces

If X is a smooth complex projective surface, it has non-trivial cohomology in degrees $0 \le r \le 4$, the dimension of the different cohomology spaces are called the Betti numbers $b_i = \dim H^i(X, \mathbb{C})$. Moreover, since X is a compact Kähler manifold, by Hodge theorem, the cohomology admits a further Hodge decomposition

$$H^r(X, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X, \mathbb{C}),$$

where the components are the (p, q)-forms on X, of dimension $h^{p,q}$. Serre duality (in this case, basically Poincare duality) yields $h^{n-p,n-q} = h^{p,q}$, and Hodge symmetry (symmetry under conjugation) yields $h^{p,q} = h^{q,p}$, yielding a Hodge diamond. (do I really want to draw?) From the Hodge diamond, we can compute other important invariants, such as

- The geometric genus $p_g = h^{0,2}$.
- The holomorphic Euler characteristic $\chi = \chi(\mathcal{O}_X) = h^{0,0} - h^{0,1} + h^{0,2}$.
- The (topological) Euler number $e(X) = c_2(X) = \sum(-1)^r b_r$.
- K^2, the self-intersection number of the canonical bundle, which is related through Noether's formula to $\chi = \frac{K^2+e}{12}$.

For a (connected) Hilbert modular surface, one can show that $h^{(1,0)} = 0$, by showing that there are no Hilbert modular forms of weight $(2,0)$, and $h^{0,0} = 1$, so the Hodge diamond degenerates (draw on previous Hodge diamond), so that $h^{(2,0)} = \chi - 1$, and $e - 2 = b_2 = 2h^{(2,0)} + h^{(1,1)}$, so the Hodge diamond can be computed completely from knowing e and K^2.

Summing up many works of many people in the 1970's and 1980's of the previous century, we have the following theorem.

Theorem 3.1 (Hirzebruch, Zagier, van der Geer, 1987)**.** *There is an (effective) algorithm to compute $K^2(\widetilde{Y}^1(\mathbb{Z}_F))$ and $e(\widetilde{Y}^1(\mathbb{Z}_F))$. Only finitely many of these surfaces are not of general type and these are completely classified (Enriques-Kodaira classification).*

For an integral ideal $\mathfrak{N} \subseteq \mathbb{Z}_F$, we may consider congruence subgroups $\Gamma(\mathfrak{N}), \Gamma_0(\mathfrak{N}), \Gamma_1(\mathfrak{N})$ as before.

Remark 3.2. For modularity applications, we are especially interested in $\Gamma_0(\mathfrak{N})$ (or $\Gamma_0^1(\mathfrak{N})$) level structure, as under an expected generalization of Serre's conjecture and a special case of the absolute Hodge conjecture, we expect that Hilbert eigenforms of level $\Gamma_0(\mathfrak{N})$ will correspond to abelian varieties over F of conductor $\mathfrak{N}$. One can see its relevance by looking at some recent works.

- Used in [Dasgupta, Kakde 2022], On the Brumer-Stark conjecture, proving it (and in fact Rubin's higher rank version of it) away from $p = 2$, leading to the resolution of the Gross-Stark conjecture.
- Used in [Loeffler, Zerbes 2020], Iwasawa theory for quadratic Hilbert modular forms, proving the main conjecture over the cyclotomic $\mathbb{Z}_p$-extension.

Remark 3.3. In some sense, this is "simply taking quotients". And yet, these specific quotients have special structure, allowing us to obtain nice formulas resulting from some beautiful math, same as in the modular curve case. Chai, in 1990, already found it missing - "Unfortunately, there is no good compactification theory for $\mathcal{M}_0(\mathfrak{f})$ ($= X_0(\mathfrak{N})$).... but they do some messy things over the supersingular locus of $\mathcal{M}$."

Theorem 3.4 (A., Babei, Breen, Costa, Duque-Rosero, Horawa, Kieffer, Kulkarni, Molnar, Schiavone, Voight, 2023)**.** *There exists an (effective) algorithm to compute the Hodge diamond of $X_0(\mathfrak{N})_\mathfrak{b}, X_1(\mathfrak{N})_\mathfrak{b}$, $X_0^1(\mathfrak{N})_\mathfrak{b}$ and $X_1^1(\mathfrak{N})_\mathfrak{b}$.*

Other than handling different types of congruences subgroup, this also automates the part of comparison between the resolution of singularities and the minimal surface. We build on that to obtain the following two theorems as well.

Theorem 3.5 (Us)**.** *Only finitely many of the Hilbert modular surfaces $X_0(\mathfrak{N})_\mathfrak{b}$ are not of general type. We list their Enriques-Kodaira classification*.*

Previous to this, other than the work of van der Geer (1987), there were some previous results of Hamahata (1994), classifying some of these surfaces with $p_g \leq 1$ for $F = \mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{p})$ where $p \equiv 1 \bmod 4$.

Theorem 3.6 (Us). *There is an (effective) algorithm to compute equations for the surfaces $Y_0^*(\mathfrak{N})_{\mathfrak{b}}$.*

This last theorem vastly generalizes past works which mainly focused on describing equations for trivial level, even until recent years.

- In [Hirzebruch, Hirzebruch-Zagier, van der Geer, up to 1987], equations for trivial level for some quadratic fields.
- In [Elkies-Kumar, 2014] constructed equations for rational and K3 Hilbert modular surfaces as double covers of the corresponding Humbert surfaces.
- In [Williams, 2020] constructed equations for Hilbert modular surfaces for some specific fields (disc 29, 37) using Borcherds products.

These have uses, e.g. in the recent work of [Cowan-Martin, 2022], constructing the moduli space of rational genus 2 curves with RM, but also in arithmetic statistics, as in [Cowan-Martin, 2023], counting abelian surfaces with RM and modular forms with small rationality fields.

4. Cusps and stabilizers

4.1. **cusp enumeration.** We use work of Dasgupta-Kakde (2021) to enumerate the cusps. If $(a : c) \in \mathbb{P}^1(F)$ represents a cusp, one notes that $I = I(a,c) = a\mathbb{Z}_F + c\mathbb{Z}_F$ is a representative of a well-defined class in $\mathrm{Cl}(F)$, and that $\mathfrak{M} = \mathfrak{M}(a,c) = \mathfrak{N} + cI^{-1}$ is a well-defined integral ideal dividing $\mathfrak{N}$. Write $(I/I\mathfrak{M})^\times$ for the set of generators of $I/I\mathfrak{M}$ as a $\mathbb{Z}_F/\mathfrak{M}$-module, and consider the orbits under the action of the squares of units $\mathbb{Z}_F^{\times 2}$ to get the set A. Similarly, starting with $(I\mathfrak{M}/I\mathfrak{N})^\times$, we obtain the set of orbits C. We then have

Theorem 4.1. *There is a natural bijection $\Gamma_0(\mathfrak{N})\backslash(F^2 \setminus \{0\}) \to \mathcal{P}_0(\mathfrak{N})$, where*

$$\mathcal{P}_0(\mathfrak{N}) = \{(I, \mathfrak{M}, a, c) : \mathfrak{M} \mid \mathfrak{N}, (a,c) \in (A \times C)/(\mathbb{Z}_F/\mathfrak{N})^\times\},$$

with the action on $A \times C$ being the anti-diagonal action.

This yields an effective algorithm to enumerate the cusps.

Example 4.2. Let $F = \mathbb{Q}(\sqrt{3})$ and $\mathfrak{p}_3 = (\sqrt{3})$, so we consider cusps in $X_0^1(\mathfrak{p}_3)$. Note that $\mathrm{Cl}(F) = \{1\}$ is trivial, so we may assume $I = \mathbb{Z}_F$. Since $\mathfrak{N} = \mathfrak{p}_3$ is prime, it has only two divisors - $\mathfrak{M} = \mathfrak{p}_3$ or $\mathfrak{M} = \mathbb{Z}_F$. Since $\mathbb{Z}_F/\mathfrak{p}_3 \simeq \mathbb{F}_3$, and $\epsilon^2 \equiv 1 \bmod \mathfrak{p}_3$, it follows that in the first case $A = \{\pm 1\}$ and C is trivial, and vice versa in the second case. Finally the anti-diagonal action of $(\mathbb{Z}_F/\mathfrak{p}_3)^\times$ on $A \times C$ identifies the two elements, so we get exactly two different cusps. It is immediate to see that $0, \infty$ are representatives for the cusps.

We then need to resolve the singularity at the cusps.

4.2. **Cusp resolution.** For that we need to understand the local behavior near the cusp, i.e. to compute the stabilizer in $\Gamma_0(N)$ of the cusp. We will see that in the case of $\Gamma_0(N)$, the stabilizer is conjugate to a group of the form

$$G(M,V) = \begin{pmatrix} V & M \\ 0 & 1 \end{pmatrix},$$

where $V \subseteq \mathbb{Z}_{F,>0}^{\times}$ is a group of totally positive units, and $M \subseteq F$ is an abelian group of rank 2 such that $VM = M$. In this case, we say that the cusp is of type $G(M,V)$.

Example 4.3. For the cusp at ∞, we see that

$$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{Z}_F^{\times}, b \in \mathbb{Z}_F \right\},$$

hence up to scalars $\Gamma_\infty \simeq G(\mathbb{Z}_F, \mathbb{Z}_F^{\times 2})$.

For the cusp at 0, the matrix $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ satisfies $\sigma 0 = \infty$, and

$$\sigma \Gamma_0 \sigma^{-1} = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{Z}_F^{\times}, b \in \mathfrak{p}_3 \right\}$$

hence up to scalars $\Gamma_0 \simeq G(\mathfrak{p}_3, \mathbb{Z}_F^{\times 2})$.

Given M, V, we may now construct $G(M,V)\backslash\mathcal{H}^2$ in two parts. First, we compute $M\backslash\mathcal{H}^2$ by embedding it in the algebraic torus $M\backslash\mathbb{C}^2 \simeq \mathbb{C}^\times \times \mathbb{C}^\times$. This isomorphism depends on the choice of basis (explicitly $e^{2\pi i \tau} = u^{m_1} v^{m_2}$), and these are related by change of basis diagrams. This then amits a natural compactification inside $\mathbb{C}^2$. Although the module M does not have a canonical basis, analysis of the local ring at ∞ leads to consider consecutive boundary points on the convex hull of $M_{>0}$ as a natural set of bases for M. Denoting these points by $\{A_k\}_{k\in\mathbb{Z}}$ each 2-dimensional cone $\sigma_k = \{sA_{k-1} + tA_k \,:\, s,t \in \mathbb{R}_{>0}\}$ leads to a copy of $\mathbb{C}^2$ with coordinates u_k, v_k, which are glued through the change of bases, to form a complex manifold. If $A_{k-1} + A_{k+1} = b_k A_k$, then the coordinate transformation from σ_k to σ_{k+1} is given by

$$(u_{k+1}, v_{k+1}) = (u_k^{b_k} v_k, u_k^{-1}).$$

The coordinate axes give rise to a sequence of non-singular rational curves S_k (defined by $v_k = 0$ and $u_{k+1} = 0$). By constructing appropriate meromorphic functions, one can show that $S_k^2 = -b_k$. Finally, the group V is cyclic, acting on $M\backslash\mathbb{C}^2$ freely and discontinuously, turning the above sequence into a cycle.

Example 4.4. (Draw!) We note that the boundary points for $\mathbb{Z}_{F,>0}$ are $A_k = ((2-\sqrt{3})^k, (2+\sqrt{3})^k)$, so that $A_{k-1} + A_{k+1} = 4A_k$, and $(2-\sqrt{3})^2 \in \mathbb{Z}_F^{\times 2}$ identifies

A_k with A_{k+2} for all k. It follows that there are two curves S_0, S_1 intersecting at two points, and each with self-intersection number -4, yielding the cusp resolution at ∞.

For $\mathfrak{p}_{3,>0}$, the boundary points are $A_0 = (3,3), A_1 = (3-\sqrt{3}, 3+\sqrt{3}), A_2 = (2-\sqrt{3})A_0$, so that $A_{-1} + A_1 = 2A_0$, $A_0 + A_2 = 3A_1$ and $(2-\sqrt{3})A_k = A_{k+2}$, so there are 4 curves in the resolution cycle with intersection numbers $(-2,-3,-2,-3)$, yielding the resolution cycle at 0.

This method has been developed by Hirzebruch. Our contribution is determining the specific M and V for a cusp given as a quadruple $(I, \mathfrak{M}, a, c)$ as above, and in the case of $\Gamma_1(\mathfrak{N})$, when it is not of this form, use finite quotients to construct the resolution.

5. Elliptic points and optimal embeddings

In general, a congruence subgroup Γ does not act freely on $\mathcal{H}^n$. The points with non-trivial finite stabilizers are called elliptic points, and they are cyclic quotient singularities. Their contribution to the invariants of the surfaces depend only on a certain invariant associated to such a point, which is called a rotation factor. Therefore, unlike the case for cusps, our goal will be only to count the number of elliptic points having a certain rotation factor.

Let $\tau = (\tau_1, \ldots, \tau_n) \in \mathcal{H}^n$ be an elliptic point, whose stabilizer is generated by $\gamma \in \Gamma$, and let $\mathrm{tr}(\gamma) = t$ and $\det(\gamma) = u$, so that $\gamma^2 - t\gamma + u = 0$, and $t^2 - 4u$ is totally negative. This already restricts us to finally many possibilities for t and u.

Example 5.1. In our running example we have $u = 1$ (since $\gamma \in \mathrm{SL}_2(\mathbb{Z})$) and $t^2 - 4 \ll 0$ means that $t \in \{0, \pm 1, \pm\sqrt{3}\}$.

The transformation $z \mapsto (z-\tau)/(z-\overline{\tau})$ maps τ to 0, and transforms γ to a rotation $\tau \mapsto \zeta\tau = (\zeta_1\tau_1, \ldots, \zeta_n\tau_n)$, so each ζ_i is a primitive root of unity with the same order as γ.

Definition 5.2. We call ζ the **rotation factor** of τ.

If τ' is another fixed point with conjugate stabilizer $\beta^{-1}\gamma\beta$, then its rotation factor satisfies $\zeta' = \zeta^{\mathrm{sgn}(\det(\beta))}$. In particular, they have the same rotation factor if and only if $\det(\beta) > 0$ is totally positive.

In order to count elliptic points, we use the observation that they correspond to embeddings of quadratic orders in quaternion orders. Indeed, let $\mathcal{O}$ be the order generated by Γ in $M_2(\mathbb{Z}_F)$. If $\gamma \in \Gamma$ is an elliptic element, then $K = F(\gamma)$ is a quadratic CM extension, and $S = \mathbb{Z}_F[\gamma] \subseteq K$ is an order, so γ induces an embbeding of $S = \mathbb{Z}_F[\gamma]$ into the order $\mathcal{O}$ and vice versa. If two such embeddings are conjugate in Γ, then their images fix the same elliptic point, so we only count these up to conjugation.

The elliptic condition ensures that there are only finitely many orders that we need consider. In order to count the number of embeddings $\phi : S \hookrightarrow \mathcal{O}$, we count them by the image $S' = \phi^{-1}(\phi(K) \cap \mathcal{O})$, so that $S' \hookrightarrow \mathcal{O}$ is **optimally embedded**.

The nice thing about optimal embeddings is that this is a local property, so we can compute the number of embeddings adelically.

Example 5.3. In our running example, for an embedding into $\mathcal{O}_0(\mathfrak{p}_3)$ to exist, a necessary condition is that there is a solution to $x^2 - tx + 1 = 0$ in $\mathbb{Z}_F/\mathfrak{p}_3 \simeq \mathbb{F}_3$. Therefore, we rule out $t = 0, t = \pm\sqrt{3}$, and remain with $t = \pm 1$. For example, consider $t = -1$, so γ is a root of x^2+x+1, i.e. $\gamma = \zeta_3$. Then $K = F(\zeta_3) = \mathbb{Q}(\sqrt{3}, i) = \mathbb{Q}(\zeta_{12})$, and $S = \mathbb{Z}_F[\zeta_3] = \mathbb{Z}_F + \sqrt{3}\mathbb{Z}_K$ (Here $\mathbb{Z}_K = \mathbb{Z}[\zeta_3, i] = \mathbb{Z}_F[i] = \mathbb{Z}[\zeta_{12}]$), so S is an order of prime conductor $\mathfrak{p}_3$, hence any embedding of S is either an optimal embedding of S or of $\mathbb{Z}_K$. But $i \in \mathbb{Z}_K$ does not embed into $\mathcal{O}$ (no elliptic points of order 2), so we only need to count optimal embeddings of S. From the fact that there is exactly one solution to $x^2 + x + 1 \equiv 0 \bmod \mathfrak{p}_3$, and that it lifts to a solution modulo $\mathfrak{p}_3^2 = 3\mathbb{Z}_F$, we see that there are 2 optimal embeddings up to conjugation (a normalized one, and its conjugate by $w_{\mathfrak{p}_3}$). The same works for $t = 1$, showing that there are exactly 4 elliptic points, all of them of order 3.

In order to count the elliptic points with a specific rotation factor, we introduce a notion of orientation for an embedding. We choose a fixed embedding $K \hookrightarrow M_2(F)$, so that any other embedding is conjugate by some $\beta \in \mathrm{GL}_2(F)$, and we say that ϕ_β is **oriented** if $\det(\beta) > 0$ is totally positive.

In order to understand the number of (conjugacy classes) of optimal embeddings $\phi : S \hookrightarrow \mathcal{O}$ of each rotation factor (corresponding to $\mathrm{sgn}(\det(\beta))$), we use a local-global principle for embeddings, and swap the roles of S and $\mathcal{O}$ - instead of moving S an see how it fits into $\mathcal{O}$, we fix S and move $\mathcal{O}$ inside its genus. Let $E = \{\beta \in B^\times : K^\beta \cap \mathcal{O} = S^\beta\}$. Then $K^\times \backslash E / \Gamma$ is in bijection with $\mathrm{Emb}(S, \mathcal{O}; \Gamma)$. We can also form the adelic analogue $\widehat{E}$, and the totally positive analog E^+. If $\mathcal{O}' \in \mathrm{Gen}\,\mathcal{O}$ is obtained by conjugation by $\widehat{\nu}$, then using strong approximation, one can show that $\mathrm{Emb}^+(S, \mathcal{O}') \neq \emptyset$ if and only if $\mathrm{nrd}(\widehat{\nu}) \in F_{>0}^\times \mathrm{nrd}(\widehat{E})$. Since it contains all the norms of elements from K, by class field theory, this is a subgroup of order at most 2 inside $\widehat{F}^\times$. Some algebra shows that the relevant group obstructing the existence of an oriented embedding is $GN^+(\mathcal{O}) = F_{>0}^\times \det(N(\widehat{\mathcal{O}}))$, and then we may form the Oriented Optimal Selectivity criterion (OOS) - K is a subfield of the corresponding class field $H_{GN^+(\mathcal{O})}$. If (OOS) does not hold, then there is no obstruction, and S embeds into all orders in the genus. In particular, all signs occur for every rotation factor. Otherwise, only half of them occur, and we can figure out which by considering the action of Frobenius.

For Eichler orders, such as $\mathcal{O}_0(\mathfrak{N})$, we have an even simpler statement for (OOS): K/F is unramified at all finite places and for every $\mathfrak{p} \mid \mathfrak{N}$ with $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{N})$ odd, $\mathfrak{p}$ splits in K.

Theorem 5.4 (Oriented Optimal Selectivity, Us)**.** *Let $\mathcal{O}$ be an Eichler order. Then*

(1) $\operatorname{Gen}\mathcal{O}$ *is orientedly optimally selective for S if and only if (OOS) holds.*
(2) *If (OOS) holds, then* $\mathrm{Emb}^+(S, \mathcal{O}') \neq \emptyset$ *for precisely half the types* $[\mathcal{O}'] \in \mathrm{Typ}^+ \mathcal{O}]$*. More precisely, if* $[\mathcal{O}'] \leftrightarrow [\mathfrak{b}] \in \mathrm{Cl}^+(R)$*, then* $\mathrm{Emb}^+(S, \mathcal{O}') \neq \emptyset$ *if and only if* $\mathrm{Frob}_{\mathfrak{b}} \in \mathrm{Gal}(K/F)$ *is trivial.*
(3) *In all cases,* $m(S, \mathcal{O}'; \mathcal{O}^\times_{>0}) = m(S, \mathcal{O}; \mathcal{O}^\times_{>0})$ *for all* $\mathcal{O}' \in \operatorname{Gen}\mathcal{O}$ *whenever both sides are nonzero.*

Corollary 5.5. *Let K/F be a CM extension, and $S \subseteq K$ an R-order. Let $\gamma \in S^\times$ be such that $\gamma R^\times \in S^\times/R^\times$ has finite non-trivial order and generates $(S^\times/R^\times)_{tor}$. Let $\mathfrak{f}$ be the conductor of S in $R[\gamma]$. Then the rotation factors $(\zeta_v^{\epsilon_v})$ which occur for fixed points of optimal embeddings of S into $O_0(\mathfrak{N})_{\mathfrak{b}}$ are exactly those with*

$$\prod_v \epsilon_v = \left(\frac{K}{\mathfrak{f}\mathfrak{b}}\right).$$

Example 5.6. In our running example, K is the narrow (ray) class field of F, hence unramified at all finite places. However, the prime $\mathfrak{p}_3$ which appears with odd valuation is inert in K, therefore (OOS) fails, showing that all possible rotation factors occur equally. Therefore, we have two elliptic points of type $(3; 1, 1)$ and two elliptic points of type $(3; 2, 1)$.

It is also possible to resolve the singularities at the elliptic points, which are cyclic quotient singularities, by considering the local ring around a cyclic quotient.

Example 5.7. For an elliptic point of type $(3; 1, 1)$ the resolution is a single (-3)-curve C, whose local canonical divisor is $K_C = \frac{1}{3}C$. For a point of type $(3; 2, 1)$ the resolution consists of two (-2)-curves, with trivial local canonical divisor.

6. Hirzebruch-Zagier divisors

Given all the data of the cusp resolutions and elliptic points, including rotation factors, we are able to compute the Euler number e and the self-intersection number K^2 using formulas originally by van der Geer. Explicitly:

$$K^2 = 2\,\mathrm{vol}(\Gamma\backslash\mathcal{H}^2) + \sum(2 - b_k) + \sum a_{q,\zeta} k_{q,\zeta}^2$$

$$e = \mathrm{vol}(\Gamma\backslash\mathcal{H}^2) + \ell + \sum a_{q,\zeta}\left(\ell_{q,\zeta} + \frac{q-1}{q}\right)$$

Example 6.1. In the example we have $\mathrm{vol}(\Gamma_0(\mathfrak{p}_3)\backslash\mathcal{H}^2) = \frac{4}{3}$, so

$$e = \frac{4}{3} + 6 + 2\left(1 + \frac{2}{3}\right) + 2\left(2 + \frac{2}{3}\right) = 16$$

and

$$K^2 = \frac{8}{3} + 2 \cdot 0 + 2 \cdot (-1) + 2 \cdot (-2) + 2 \cdot \left(-\frac{1}{3}\right) = -4.$$

We deduce the Hodge diamond - $\chi = 1$, hence $p_g = 0$, and $h^{1,1} = 16$.

This is not enough for classification since this surface is not minimal. However, we can use another idea - Hirzebruch-Zagier divisors. These divisors are modular curves and Shimura curves that live on our surface (an obvious example is the image of the diagonal (z, z)). When these are curves with small levels, they are often rational, leading to exceptional curves that we can blow down.

Example 6.2. Consider F_3, the image of $(z, 3z)$ on our surface. A simple calculation ($K \cdot F_3 = \frac{4}{3} - 2 - \frac{1}{3} = -1$) shows that F_3 is a (-1)-curve isomorphic to $X_0(3)$, which intersects the cusp resolutions at the 2 (-2) curves above 0. Blowing it down leads to a pair of intersecting exceptional curves, showing that our surface is rational. (all plurigenera vanish - intersect a holomorphic section with $C_1 + C_2$).

This should not come as a surprise, given work of Bruin-Flynn-Shnidman (2021) giving an explicit rational parametrization of genus 2 curves with full $\sqrt{3}$-level structure on their Jacobians.

7. Summary